Exact Solution of Photon Equation in Stationary Gödel-type and Gödel Space-Times

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Abstract

In this work the photon equation (massless Duffin-Kemmer-Petiau equation) is written expilicitly for the general type of stationary Gödel space-times and is solved exactly for the Gödel-type and the Gödel space-times. Harmonic oscillator behaviour of the solutions is discussed and energy spectrum of photon is obtained.

Keywords: Duffin-Kemmer-Petiau equation, Exact Solution, Gödel space-times

I. INTRODUCTION

The relativistic particles satisfying wave equations in cosmological backgrounds are considered to analyze quantum effects in curved space-times. For this purpose single particle states are studied in detail and also some restrictions on solutions are examined to obtain behavior of the particles. In the atomic scale relativistic wave equations which conformed to general relativity may not be important because of the weakness of gravitational effects. On the contrary for many astrophysical situations one has to take into account gravitational effects due to it's dominant role; for example particle creation by black holes.

The most studied relativistic equations are Klein-Gordon and Dirac equations describing spin zero and spin one-half particles respectively. These equations are considered mostly in the expanding universes [1–6] which are members of Friedmann cosmological models. Other type of universes are so-called Gödel and Gödel-type space-times where they have a global rotation in all point of the space. Theoretical works of Gamow [7] and Gödel [8], and some other evidences of rotation [9–11] are directed us to study the behavior of electromagnetic fields [12–15] and spinning particles in curved rotating backgrounds. The line-elements of the Gödel space-time

$$ds^2 = -(dt + e^{ar}d\theta)^2 + dr^2 + \frac{1}{2}(e^{ar}d\theta)^2 + dz^2, \tag{1}$$

and the Gödel-type space-time

$$ds^2 = -(dt + md\theta)^2 + dr^2 + (l + m^2)d\theta^2 + dz^2 \tag{2}$$

are studied for Klein-Gordon and Weyl equations [16–19]. The functions m and l in the Gödel-type line-element are only functions of r and they satisfy following two independent conditions [15]:

$$D = \left(l + m^2\right)^{1/2} = A_1 \exp(ar) + A_2 \exp(-ar), \tag{3}$$

$$\frac{1}{D}\frac{dm}{dr} = C \tag{4}$$

or

$$D = Ar, \frac{1}{D}\frac{dm}{dr} = C, \tag{5}$$

where A_1, A_2, A and C are arbitrary constants. Second condition gives the singular homogeneous Gödel-type solution, but it can not be obtained from the first condition. As a particular choice of the first condition one takes $a = 0$, then

$$D = const. \tag{6}$$

If we choose $A_1 = 1/\sqrt{2}$ and $A_2 = 0$, then $D = (1/\sqrt{2})\exp(ar)$. This gives $m = (C/\sqrt{2}a)\exp(ar)$ (with a constant of integration equal to zero) and if we take $C = -a\sqrt{2}$, then $m = \exp(ar)$ and $l = (-1/2)\exp(2ar)$, that gives Gödel solution (with $C < 0$) [8].

Although there are some works concerning with the electromagnetic fields in these space-times there is not any attempt to find exact solution of the quanta of electromagnetic field, i.e. photon. The photon is a massless spin-one particle and it obeys massless case of Duffin-Kemmer-Petiau (DKP) equation which can be formed as

$$\left[i\beta^{(\alpha)}\partial_{(\alpha)} + m\right]\Psi = 0 \tag{7}$$

where $\beta^{(\alpha)} = \gamma^{(\alpha)}\otimes\mathrm{I}+\mathrm{I}\otimes\gamma^{(\alpha)}$. This 16×16 matrix equation is in Minkowski frame and it must be conformed to general relativity to find out quantum effects in curved backgrounds. In the studies of Ünal [20] and Lunardi et al. [21] it has been shown that the covariant form of DKP equation is given by

$$(i\beta^{\mu}\nabla_{\mu} + m)\Psi = 0 \tag{8}$$

where $\beta^{\mu}(x) = \gamma^{\mu}(x)\otimes\mathrm{I} + \mathrm{I}\otimes\gamma^{\mu}(x)$ are the Kemmer matrices in curved space-time and they are related to flat Minkowski space-time as

$$\beta^{\mu} = e^{\mu}_{(\alpha)}\beta^{(\alpha)} \tag{9}$$

with a tetrad frame that satisfies

$$g_{\mu\nu} = e^{(\alpha)}_{\mu}e^{(\beta)}_{\nu}\eta_{(\alpha)(\beta)}. \tag{10}$$

The covariant derivative in Eq.(8) is

$$\nabla_{\mu} = \partial_{\mu} + \Sigma_{\mu}, \tag{11}$$

with spinorial connections which can be written as

$$\Sigma_{\mu} = \Gamma_{\mu}\otimes\mathrm{I} + \mathrm{I}\otimes\Gamma_{\mu}, \tag{12}$$

where

$$\Gamma_{\rho} = \frac{1}{8}\left[\tilde{\gamma}^{(i)}, \tilde{\gamma}^{(k)}\right] e^{\nu}_{(i)}e_{(k)\nu;\rho}. \tag{13}$$

In the massless limit of DKP equation the particle-antiparticle are identical and hence mass eigenvalue is zero. Therefore, DKP equation reduces to 4×4 massless DKP equation as follow [20]

$$\beta^{\mu}\nabla_{\mu}\Psi = 0, \tag{14}$$

where β^{μ} are now

$$\beta^{\mu}(x) = \sigma^{\mu}(x) \otimes \mathrm{I} + \mathrm{I} \otimes \sigma^{\mu}(x) \tag{15}$$

with $\sigma^{\mu}(x) = (\mathrm{I}, \overrightarrow{\sigma}(x))$ and

$$\nabla_{\mu} = \partial_{\mu} + \Sigma_{\mu}, \tag{16}$$

where spinorial connections Σ_{μ} are given with the limit $\gamma^{\mu} \to \sigma^{\mu}$ as

$$\Sigma_{\mu} = \lim_{\gamma\to\sigma} \Gamma_{\mu} \otimes \mathrm{I} + \mathrm{I} \otimes \Gamma_{\mu}. \tag{17}$$

In this paper we study solution of the photon equation (14) in a singular homogeneous stationary Gödel-type and a stationary Gödel universes. In Section II we write down four coupled equations for the general stationary Gödel-type universes and then solved for singular homogeneous Gödel-type universe. In Section III the equations obtained in Section II is considered for Gödel universe and solved. In Section IV both results are discussed and quantum mechanical oscillatory regions are found. Finally energy spectrums are obtained.

II. SOLUTION OF PHOTON EQUATION FOR THE GÖDEL-TYPE UNIVERSE

For the line-element given Eq.(2) it must be introduced tetrads with a suitable selection. For the simplicity we can choose

$$e^{\mu}_{(0)} = \delta^{\mu}_{0}, e^{\mu}_{(1)} = \delta^{\mu}_{1}, e^{\mu}_{(3)} = \delta^{\mu}_{3}, e^{\mu}_{(2)} = \frac{1}{D}(\delta^{\mu}_{2} - m\delta^{\mu}_{0}) \tag{18}$$

where $x^0 = t, x^1 = r, x^2 = \theta, x^3 = z$ [17].

The curved Dirac matrices which satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \tag{19}$$

are given by $(\gamma^{\mu}(x) = e^{\mu}_{(\alpha)}\widetilde{\gamma}^{(\alpha)})$

$$\gamma^0 = \widetilde{\gamma}^0 - \frac{m}{D}\widetilde{\gamma}^2, \gamma^1 = \widetilde{\gamma}^1, \gamma^2 = \frac{1}{D}\widetilde{\gamma}^2, \gamma^3 = \widetilde{\gamma}^3. \tag{20}$$

The spinorial connections are

$$\Gamma_0 = \frac{m'}{4D}\widetilde{\gamma}^2\widetilde{\gamma}^1, \Gamma_1 = \frac{m'}{4D}\widetilde{\gamma}^0\widetilde{\gamma}^2, \Gamma_2 = -\frac{m'}{4}\widetilde{\gamma}^1\widetilde{\gamma}^0 + \frac{mm' + l'}{4D}, \Gamma_3 = 0 \tag{21}$$

where prime indicates derivative with respect to r. Using Eq.(14) and Eq.(15) and Jauch and Rohrlich [23] representation of Dirac matrices we obtain the photon equation as

$$\left\{\left[-2i(\mathrm{I}\otimes\mathrm{I}) - \frac{m}{D}(\widetilde{\sigma}^2 \otimes \mathrm{I} + \mathrm{I} \otimes \widetilde{\sigma}^2)\right]\partial_t + (\widetilde{\sigma}^1 \otimes \mathrm{I} + \mathrm{I} \otimes \widetilde{\sigma}^1)\partial_r + \frac{1}{D}(\widetilde{\sigma}^2 \otimes \mathrm{I} + \mathrm{I} \otimes \widetilde{\sigma}^2)\partial_\theta\right.$$

$$+(\tilde{\sigma}^3 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^3)\left(\partial_z - \frac{m'}{2D}\right) + \frac{iD'}{2D}(\tilde{\sigma}^2 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^2)(\tilde{\sigma}^3 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^3)$$

$$-\frac{im'}{4D}\left[(\tilde{\sigma}^1 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^1)(\tilde{\sigma}^2 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^2) - (\tilde{\sigma}^2 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^2)(\tilde{\sigma}^1 \otimes \mathrm{I} + \mathrm{I} \otimes \tilde{\sigma}^1)\right]\Bigg\}\Psi = 0. \tag{22}$$

To obtain stationary solutions it is used a function as follows

$$\Psi = \exp\left[-i(\omega t + k_2\theta + k_3 z)\right]\Phi, \tag{23}$$

hence Eq.(22) gives four coupled differential equations in terms of the components of the spinor as

$$\partial_r(\Phi_B + \Phi_C) + \frac{1}{D}(m\omega - k_2)(\Phi_B + \Phi_C) - 2(\omega + ik_3)\Phi_A = 0, \tag{24}$$

$$\left(\partial_r - \frac{D'}{D}\right)(\Phi_A + \Phi_D) - \frac{1}{D}(m\omega - k_2)(\Phi_A - \Phi_D) - 2\omega\Phi_B = 0, \tag{25}$$

$$\left(\partial_r - \frac{D'}{D}\right)(\Phi_A + \Phi_D) - \frac{1}{D}(m\omega - k_2)(\Phi_A - \Phi_D) - 2\omega\Phi_C = 0, \tag{26}$$

$$\partial_r(\Phi_B + \Phi_C) - \frac{1}{D}(m\omega - k_2)(\Phi_B + \Phi_C) - 2(\omega - ik_3)\Phi_A = 0. \tag{27}$$

From Eq.(25) and Eq.(26) $\Phi_B = \Phi_C$. Up to now we have not used an explicit form of $m(r)$ and $l(r)$. Hence, different choices of m and l lead us to different rotating models.

For the Eq.(5) we have that $m = m_0 + CAr^2/2$ and this is the singular homogeneous Gödel-type universe and Eqs.(24)-(27) reduces to

$$[\partial_r + \frac{1}{r}(\tilde{\omega} + \beta r^2)]\Phi_B - \frac{1}{\alpha}\Phi_A = 0, \tag{28}$$

$$\left(\partial_r - \frac{1}{r}\right)(\Phi_A + \Phi_D) - \frac{1}{r}(\tilde{\omega} + \beta r^2)(\Phi_A - \Phi_D), \tag{29}$$

$$[\partial_r - \frac{1}{r}(\tilde{\omega} + \beta r^2)]\Phi_B - \frac{1}{\alpha^*}\Phi_D = 0, \tag{30}$$

where

$$\tilde{\omega} = \frac{m_0\omega}{A} - \frac{k_2}{A}, \beta = \frac{C}{2}, \alpha = (\omega + ik_3)^{-1}. \tag{31}$$

Eliminating Φ_A and Φ_D from Eq.(28) and Eq.(30) we obtain following second order differential equation for Φ_B as

$$\partial_r^2\Phi_B - \frac{1}{r}\partial_r\Phi_B + \left(-\frac{H}{r^2} - E - \beta^2 r^2\right)\Phi_B = 0, \tag{32}$$

where

$$E = \widetilde{\omega}C + \omega^2 + k_3^2, H = \widetilde{\omega}^2 - \frac{2\sqrt{2}ik_3\widetilde{\omega}}{\omega}. \tag{33}$$

Introducing a variable as $u = \beta r^2$ it is found that

$$4u^2\partial_u^2\Phi_B + \left(-u^2 - H - \frac{E}{\beta}u\right)\Phi_B = 0, \tag{34}$$

where if it is used for constants $H = 4\lambda^2 - 1$ and $4\kappa = -E/\beta$, one can be obtained well-known Whittaker differential equation has solution [24]

$$\Phi_B(u) = B_1 W_{\kappa,\lambda}(u) + B_2 M_{\kappa,\lambda}(u). \tag{35}$$

III. SOLUTION OF PHOTON EQUATION FOR THE GÖDEL UNIVERSE

As was mentioned before the Gödel space-time can be obtained from the Gödel-type space-time by taking $m = \exp(ar)$ and $l = (-1/2)\exp(2ar)$ in Eq.(2). So it is sufficient to introduce these values of m and l in Eqs.(24)-(27) to obtain solution of photon equation for the Gödel space-time. Then we obtain following coupled equations

$$\partial_r\Phi_B + \sqrt{2}(\omega - k_2 e^{-ar})\Phi_B - \frac{1}{\alpha}\Phi_A = 0, \tag{36}$$

$$(\partial_r - a)(\Phi_A + \Phi_D) - \sqrt{2}(\omega - k_2 e^{-ar})(\Phi_A - \Phi_D) - 2\omega\Phi_B = 0, \tag{37}$$

$$\partial_r\Phi_B - \sqrt{2}(\omega - k_2 e^{-ar})\Phi_B - \frac{1}{\alpha^*}\Phi_D = 0, \tag{38}$$

where again $\Phi_B = \Phi_C$. Substituting Eq.(36) and Eq.(38) in Eq.(37) it is found second order differential equation as follows

$$\partial_r^2\Phi_B + \left(-2k_2^2 e^{-2ar} + Ae^{-ar} - B\right)\Phi_B = 0, \tag{39}$$

where

$$A = 4\omega - \frac{i2\sqrt{2}ak_3}{\omega}, B = 3\omega^2 + k_3^2 - i\sqrt{2}k_3 a. \tag{40}$$

If variable is changed as $u = \frac{\sqrt{8}k_2 e^{-ar}}{a}$ and a function $\Phi_B = u^{-1}g_B$ is used, the following Whittaker differential equation is obtained

$$4u^2\partial_u^2 g_B + [-u^2 + 4\sigma u - (4\rho^2 - 1)]g_B = 0. \tag{41}$$

The solution of the Eq.(41) can be written as

$$g_B(u) = C_1 W_{\sigma,\rho}(u) + C_2 M_{\sigma,\rho}(u) \tag{42}$$

with

$$4\sigma = \frac{\sqrt{2}A}{a}\, , \rho^2 = \frac{B}{4a^2} + \frac{1}{4}. \tag{43}$$

IV. EXACT SOLUTIONS

The solutions Eq.(35)and Eq.(42) can be written in terms of the confluent hypergeometric functions U and M using

$$W_{\alpha,\xi}(x) = e^{-x/2}x^{\xi+1/2}U(\frac{1}{2} + \xi - \alpha, 1 + 2\xi; x), \tag{44}$$

$$M_{\alpha,\xi}(u) = e^{-x/2}x^{\xi+1/2}M(\frac{1}{2} + \xi - \alpha, 1 + 2\xi; x), \tag{45}$$

so exact solution of photon equation for the singular homogeneous Gödel-type universe is

$$\Psi_B = e^{-i(\omega t + k_2\theta + k_3 z)}e^{-(\frac{Cr^2}{4})}(\frac{Cr^2}{2})^{\lambda+1/2}[B_1 U(\frac{1}{2} + \lambda - \kappa, 1 + 2\lambda; \frac{Cr^2}{2})$$

$$+ B_2 M(\frac{1}{2} + \lambda - \kappa, 1 + 2\lambda; \frac{Cr^2}{2})]. \tag{46}$$

Similarly solution of the photon equation for the Gödel universe is

$$\Psi_B = (\frac{\sqrt{8}k_2 e^{-ar}}{a})^{\rho-1/2}e^{-i(\omega t + k_2\theta + k_3 z)}e^{-(\frac{\sqrt{8}k_2 e^{-ar}}{2a})}$$

$$\times [C_1 U(\frac{1}{2} + \rho - \sigma, 1 + 2\rho; \frac{\sqrt{8}k_2 e^{-ar}}{a}) + C_2 M(\frac{1}{2} + \rho - \sigma, 1 + 2\rho; \frac{\sqrt{8}k_2 e^{-ar}}{a})]. \tag{47}$$

From the condition on Whittaker functions that must be bounded for all values of variable we find energy spectrum for both space-times as follows

$$\frac{1}{2} + \lambda - \kappa = -n_{1,} \tag{48}$$

$$\frac{1}{2} + \rho - \sigma = -n_2. \tag{49}$$

where n_1 and n_2 are positive integers or zero. The explicit forms of Eqs.(48) and (49) are

$$n_1 = -\frac{\omega^2}{2C} - \frac{k_3^2}{2C} - \frac{\widetilde{\omega}}{2} \mp \sqrt{\widetilde{\omega}^2 - 2\sqrt{2}ik_3\frac{\widetilde{\omega}}{\omega} + 1} - \frac{1}{2}, \tag{50}$$

$$n_2 = \frac{\sqrt{2}\omega}{a} \mp \frac{1}{2a}\sqrt{3\omega^2 + k_3^2 - ia\sqrt{2}k_3 + a^2} - \frac{ik_3}{\omega} - \frac{1}{2}. \tag{51}$$

One can show that solutions (46) and (47) are oscillatory in behavior only regions (for detail Ref. [12])

$$-\frac{1}{2}(\omega^2 + \widetilde{\omega}C + k_3^2) - \sqrt{2(\omega^2 + \widetilde{\omega}C + k_3^2) - \frac{\widetilde{\omega}^2 C}{2} + \sqrt{2}i\frac{k_3\widetilde{\omega}C}{\omega}} < u <$$

$$-\frac{1}{2}(\omega^2 + \widetilde{\omega}C + k_3^2) + \sqrt{2(\omega^2 + \widetilde{\omega}C + k_3^2) - \frac{\widetilde{\omega}^2 C}{2} + \sqrt{2}i\frac{k_3\widetilde{\omega}C}{\omega}}, \tag{52}$$

$$\omega - \frac{iak_3}{\sqrt{2}\omega} - \sqrt{(\omega - \frac{iak_3}{\sqrt{2}\omega})^2 - \frac{3\omega^2}{2} - \frac{k_3^2}{2} + \frac{iak_3}{\sqrt{2}}} < k_2 u <$$

$$\omega - \frac{iak_3}{\sqrt{2}\omega} + \sqrt{(\omega - \frac{iak_3}{\sqrt{2}\omega})^2 - \frac{3\omega^2}{2} - \frac{k_3^2}{2} + \frac{iak_3}{\sqrt{2}}}. \tag{53}$$

V. RESULTS AND DISCUSSIONS

We have analyzed the photon equation in the background of stationary Gödel-type and the Gödel universes. The method of separation of variables is used because of the simple symmetry of the Gödel universes. The energy spectrum of particle and oscillatory character of solutions were found.

The results obtained can be used to study quantum field theory in curved rotating space-times. Also it is necessary to quantize obtained wave functions to discuss pair creation and annihilation. In addition, these results can be compared with earlier results of electromagnetic fields in Gödel space-times and calculate quantum corrections on these results. The effects of rotation on propagation and helicity of electromagnetic fields can be obtained by using the wave functions found in the Section II and Section III.

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